now endow him with sense organs in all other respects similar to our own, and also permit him to move about, the notion of a point would also present itself to him, in a secondary way however, and brought to his attention principally by the sense of touch. He would observe that the line image of a point rotates in a certain way when he inclines his head to the right or left; so that, by associating this fact with the idea of a point obtained by the sense of touch, he would come to look upon a point as that geometric configuration which is common to all of the lines which pass through it. He would analyze the point into line-elements, just as we analyze the line into point-elements.

We have indicated the existence of two distinct four-dimensional geometries without leaving the realm of ordinary space. But the question may be asked: is a space conceivable which, considered as an aggregate of *points*, has four dimensions? Probably it is this phase of the question of which most persons think when the subject of the fourth dimension comes under discussion.

That such a space is conceivable in a *logical, purely mathematical sense* is evident from what has been said. Abstractly, *i. e.*, without reference to concrete sense impressions, a point may be defined as any concept which is determined by a set of numbers. We, therefore, obtain at once a four-dimensional point-space in the abstract sense, by choosing the space-element of ordinary space in such a way as to give rise to a four-dimensional geometry, and then substituting the word *point* for the word *space-element*. If it be desired, however, to obtain a more concrete illustration to show the possibility of the *objective existence* of a four dimensional point-aggregate, we shall be compelled to analyze somewhat more closely the psychological foundations of our space concept. This will lead to some speculations which, while highly imaginative, are not illogical.

It is a familiar notion that those properties of things which we call tone and color do not exist objectively. The vibrations of the air or of the ether, converted into nervous energy and transmitted to the brain, produce upon the mind the peculiar impressions of tone and color. Many philosophers claim that space, likewise, has no objective existence. The spacial relation, according to them, is to be looked upon as something which the mind creates in reacting upon the stimuli of the external world; the mind creates space as it does tone and color. If this is true, space is for us such as we see it; not because it *is* so, but because we *see* it so. If there existed a race of men, or even a single rational being whose space concept involved more than three dimensions, that new kind of space would have the same claim to existence as the old.

Since our argument clearly becomes meaningless on this basis, let us take the point of view that an absolute space exists, and that it has intrinsic properties which are not created by the mind. We shall attempt to show that, even on that assumption, beings somewhat differently organized from

ourselves could develop a space concept vastly different from our own. One such case has already been indicated.

The mere fact that we find it difficult to conceive of what the fourth dimension would "look like," no more disproves its existence than the inability of a blind man to conceive of color can be held to show that no such thing exists. It is not hard to imagine a race of beings (flat landers), living in a plane, to whom a third dimension might appear as mysterious as a fourth does to us. Suppose that such a flatlander were to surround an identifiable object of his infinitely thin world by a circle. Nothing can be more obvious to him than the proposition that this object can never reach the outside of the circle without crossing the circumference. But suppose that he sees it disappear, and reappear a moment later outside. To us there is nothing mysterious about this; the object has described a path in space. The flatlander, however, would have to disbelieve his senses, call to his assistance the supernatural or adopt the hypothesis of a three-dimensional space, of which his own two-dimensional world is but a part. It has been proposed to explain some of the phenomena alleged to have been observed by the spiritualists on the assumption that space has four dimensions. A four dimensional being, for instance, could easily escape from a hermetically closed room. For him, a sphere is no more effective as a barrier than is a circle for us. We see then the possibility of obtaining experimental evidence for the objective existence of a fourth dimension, without wishing to assert that such evidence has already been furnished.

But we may look at the question in still another way. It can hardly be denied that our space concept would be much modified if our sense organs were different from what they are. What, in fact, are the experiences upon which our own ideas of space are based? Obviously the sense of sight is an important factor. It is clear, however, that our geometry would be two dimensional if we were stationary beings, rooted to the ground like a tree, provided with but a single immobile eye in focus for all distances, or if the act of accommodation were not accompanied by muscular sensations. We should be deprived of the notion of distance. The existence of an arm, an organ of touch, etc., enables us to enlarge our idea of space so as to include the notion of distance. Our space concept, then, is an abstraction which unites for us in a consistent manner, a number of qualitatively different sense impressions.

But we do not unite all of our sense-impressions in this way. The notion of color, for instance, does not differ qualitatively from that of direction even as much as does the notion of distance, being a result of pure sight impressions, while that of distance is not. It is quite conceivable that color is no more an intrinsic property of an object than is its distance from us. It is not absurd to imaginine that bodies are actually immersed in a space of four dimensions, but that only three of these present themselves to us as such, the fourth manifesting itself to us as color. We can imagine a being

so organized that differences of color, *i. e.*, differences in the rapidity of ether vibrations, would present themselves to him somewhat as distances from a fixed plane do to us. His intuition would, in a sense, be closer to the truth than ours. For, he would see at a glance that the differences which we attribute to color are relations of a quantitative rather than of a qualitative nature, an idea at which *we* can arrive only as a consequence of elaborate theories and complicated experiments.

Without asserting such to be the case, we cannot dismiss offhand the possibility that absolute space, if it exists at all, has more than three dimensions, that these additional dimensions do not however present themselves to us as such, but become sensible in a masked form as color, electric charges, etc. There is no logical reason why a mathematical theory of color or of electricity might not be built upon such a basis.

Unless we assume dogmatically that space is exactly as it appears to us because we create it with all of its properties, so that objectively it does not exist at all, and unless we deny, dogmatically, the possibility that some other race of thinking beings might possess a space concept also subjective, but different from ours, there remains only one possible way of saving the proposition that space cannot have more than three dimensions, and that And that would be the trivial method of defining it in that way.

ON THE NUMERICAL FACTORS OF CERTAIN ARITHMETIC FORMS.*

By R. D. CARMICHAEL, Princeton University.

The methods used by Dickson in an article on the cyclotomic function† may be applied with some change so as to obtain a set of more general propositions than those at which he arrives and of such nature that his theorems are the simplest cases of those thus obtained. The object of this paper is to generalize the theorems of Dickson; and his method of proof will be freely employed.

We shall let

$$Q_n(x) = 0$$

be the equation whose roots are the primitive nth roots of unity without repetition. Then $Q_n(x)$ is a polynomial in x with integral coefficients, that of the highest power being unity. The degree of the equation is $\phi(n)$, $\phi(n)$ being Euler's function.

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*Read before the American Mathematical Society, February 29, 1908.

†THE AMERICAN MATHEMATICAL MONTHLY, Vol. XII, pp. 86–89.

As a permanent notation, set

$$n=p_1^{a_1}\, p_2^{a_2}\ldots p_r^{a_r}$$

where the p's are different primes and the a's are integers. From the theory of the primitive roots of unity, we have*

$$(1) \qquad x^n-1=\Pi\, Q_d(x),$$
$$(2) \qquad x^{n/p_1}-1=\Pi\, Q_\delta\,(x),$$

where d ranges over all the divisors of n, and δ over all the divisors of n/p_1. Dividing equation (1) by equation (2), member for member, we have

$$(3) \qquad x^{n(p_1-1)/p_1}+x^{n(p_1-2)/p_1}+\ldots+x^{n/p_1}+1=\Pi\, Q_c(x),$$

where c ranges over all the divisors of n containing the factor $p_1^{a_1}$. Hence $Q_c(x)$ is a polynomial in x with integral coefficients.

Let β be any integer positive or negative, zero excluded, and α any integer prime to β. (If $\beta=1$, α is any integer whatever.) In (3) replace x by α/β. Multiply both terms of the resulting equation by $\beta^{n(p_1-1)/p_1}$; we have

$$(4) \qquad \begin{aligned}&\alpha^{n(p_1-1)/p_1}+\alpha^{n(p_1-2)/p_1}\,\beta^{n/p_1}+\alpha^{n(p_1-3)/p_1}\,\beta^{2n/p_1}+\ldots\\ &\quad+\alpha^{n/p_1}\,\beta^{n(p_1-2)/p_1}+\beta^{n(p_1-1)/p_1}\\ &\quad=\beta^{n(p_1-1)/p_1}\,\Pi\, Q_c(\alpha/\beta).\end{aligned}$$

Denote by $Q_c(\alpha, \beta)$ the quantity $\beta^{\phi(c)}Q_c(\alpha/\beta)$. It is evident that $Q_c(\alpha, \beta)$ is a homogeneous polynomial in α, β with integral coefficients. Equation (4) may now take the following form:

$$(5) \qquad \alpha^{n(p_1-1)/p_1}+\alpha^{n(p_1-2)/p_1}\,\beta^{n/p_1}+\ldots+\alpha^{n/p_1}\,\beta^{n(p_1-2)/p_1}+\beta^{n(p_1-1)/p_1}=\Pi\, Q_c(\alpha,\ \beta).$$

In $x^{n/p_1}-1$ replace x by α/β and multiply by β^{n/p_1}. This gives $\alpha^{n/p_1}-\beta^{n/p_1}$. We are now able to find the highest common factor of $\alpha^{n/p_1}-\beta^{n/p_1}$, and $Q_c(\alpha, \beta)$. To this end divide the first member of (5) by $\alpha^{n/p_1}-\beta^{n/p_1}$. A remainder of $p_1.\beta^{n(p_1-1)/p_1}$ is found. This remainder must contain the greatest common divisor sought. But since α and β are relatively prime $\alpha^{n/p_1}-\beta^{n/p_1}$ is not divisible by β. Therefore 1 or p_1 is the greatest common divisor of $\alpha^{n/p_1}-\beta^{n/p_1}$ and $\Pi Q_c(\alpha, \beta)$, the second member of (5). Therefore only one factor of $\Pi Q_c(\alpha, \beta)$ can contain p_1 if $\alpha^{n/p_1}-\beta^{n/p_1}$ contains p_1^2. Hence we have the following theorem:

*See Bachmann's *Kreistheilung*, especially the third and fifth lectures.

THEOREM I. *When α and β are relatively prime integers, the greatest common divisor of $\alpha^{n/p_1}-\beta^{n/p_1}$ and any $Q_c(\alpha, \beta)$ is 1 or p_1, n and c being defined as before. Also, not more than one of the numbers $Q_c(\alpha, \beta)$ contains the factor p_1 when $\alpha^{n/p_1}-\beta^{n/p_1}$ contains p_1^2.*

Suppose that $\alpha^{n/p_1}-\beta^{n/p_1}$ is divisible by p_1, and set this number equal to kp_1, where k is an integer. Then

$$\alpha^{n/p_1}=\beta^{n/p_1}+kp_1.$$

Substituting this value of α^{n/p_1} in the left member of (5) and combining like powers of β, we have

$$p_1\beta^{n(p_1-1)/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\beta^{n(p_1-2)/p_1}+\text{terms in } p_1^2,\ p_1^3,\ \ldots$$

Therefore p_1^2 divides the first member of (5) only when

$$p_1\beta^{n(p_1-1)/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\beta^{n(p_1-2)/p_1}\equiv 0 \pmod{p_1^2}.$$

Now p_1 is a factor of $\alpha^{n/p_1}-\beta^{n/p_1}$, a number to which β is evidently prime; for α and β are relatively prime integers. Hence β is not divisible by p_1. Then the above congruence reduces to

$$p_1\beta^{n/p_1}+\tfrac{1}{2}p_1(p_1-1)kp_1\equiv 0 \pmod{p_1^2}. \tag{6}$$

It is evident that this does not hold when $p_1>2$. Hence the second member of (5) does not contain p_1^2 when $\alpha^{n/p_1}-\beta^{n/p_1}$ contains $p_1>2$. Hence

THEOREM II. *When α and β are relatively prime integers and p_1 is an odd prime dividing $\alpha^{n/p_1}-\beta^{n/p_1}$, not more than one of the numbers $Q_c(\alpha, \beta)$, c as before, is divisible by p_1, and none of them is divisible by p_1^2.*

Consider the case $p_1=2$ in congruence (6) above. We have

$$2\beta^{n/2}+2k\equiv 0 \pmod 4,$$

which reduces to

$$\beta^{n/2}+k\equiv 0 \pmod 2.$$

Hence β and k are both odd or both even. Now the equation $\alpha^{n/p_1}-\beta^{n/p_1}=kp_1$ reduces in the present case to

$$\alpha^{n/2}-\beta^{n/2}=2k. \tag{7}$$

If k and β are both even, then (7) shows that α is also, contrary to the as-

sumption of α and β relatively prime. Hence k and β are both odd; and therefore by (7) α is also. Moreover it is evident from (7) that of the numbers α^{n+2}, β^{n+2} one is congruent to 1 and the other to 3 modulo 4. This requires that $n/2$ shall be odd and that of the numbers α and β one shall be congruent to 1 and the other to 3 modulo 4.

Since $n/2$ is odd it follows that the range of c is over a set of numbers each of which is twice an odd number, unity being included among these odd numbers. u being such an odd number it is evident that $Q_{2u}(\alpha, \beta)$ is equal to or a factor of $\alpha^{u-1}+\alpha^{u-2}\beta+\ldots+\beta^{u-1}$, when $u>1$. But since neither α nor β is even, this last expression contains an odd number of odd terms and is therefore odd. Hence $Q_{2u}(\alpha, \beta)$ is odd when $u>1$. Also, $Q_2(\alpha, \beta)=\alpha+\beta$ and is now divisible by 4; for $\alpha+\beta\equiv 1+3 \pmod 4$. From these considerations we deduce the following proposition:

THEOREM III. *If α and β are relatively prime integers and 2 is a factor of $\alpha^{n/2}-\beta^{n/2}$, then $Q_c(\alpha, \beta)$ is divisible by 2^2 only when $c=2$ and of the numbers α, β one is congruent to 1 and the other to 3 modulo 4.*

We shall now determine the case in which $Q_c(\alpha, \beta)$ is divisible by p_1.

Set $c=mp_1^{a_1}$, m being an integer >1 and not divisible by p_1. Dickson has shown* that

$$Q_c(x)=Q_n(x^{p_1^{a_1}}) \div Q_m(x^{p_1^{a_1-1}}).$$

If in this equation x is replaced by α/β and each function Q is multiplied by that power of β whose index is equal to the degree of the function, there results the following equation:

$$(8) \qquad Q_c(\alpha, \beta)=Q_m(\alpha^{p_1^{a_1}}, \beta^{p_1^{a_1}}) \div Q_m(\alpha^{p_1^{a_1-1}}, \beta^{p_1^{a_1-1}}).$$

Now by Fermat's Theorem, $\alpha^{p_1}\equiv\alpha$, $\beta^{p_1}\equiv\beta \pmod{p_1}$. Therefore the second member of (8) is congruent to

$$Q_m(\alpha, \beta) \div Q_m(\alpha, \beta)$$

modulo p_1. Now this quantity is 1 unless $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$. Hence $Q_c(\alpha, \beta)\equiv 1 \pmod{p_1}$ unless $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$.

Consider the case when $Q_m(\alpha, \beta)\equiv 0 \pmod{p_1}$. Now $Q_m(\alpha, \beta)$ divides algebraically

$$(9) \qquad (\alpha^m-\beta^m) \div (\alpha^{m/q}-\beta^{m/q})=\alpha^{m(q-1)/q}+\alpha^{m(q-2)/q}\beta^{m/q}+\ldots+\beta^{m(q-1)/q},$$

where q is any prime factor of m. If

*THE AMERICAN MATHEMATICAL MONTHLY, Vol. XII, p. 86.

(10) $$\alpha^{m/q} \equiv \beta^{m/q} \pmod{p_1},$$

the second member of (9) is congruent to $q\,\beta^{m(q-1)/q} \pmod{p_1}$. Then there is some integer k such that

$$kQ_m(\alpha, \beta) \equiv q\,\beta^{m(q-1)/q} \pmod{p_1}.$$

Now q is different from p_1; for it is a factor of m which is prime to p_1. Therefore when (10) holds $Q_m(\alpha, \beta)$ is not divisible by p_1 when β is not so divisible. But if β is divisible by p_1, $Q_m(\alpha, \beta)$ is not congruent to zero modulo p_1; and hence $Q_c(\alpha, \beta) \equiv 1 \pmod{p_1}$.

There is left the case $\alpha^{m/q}$ not congruent to $\beta^{m/q}$ and $Q_m(\alpha, \beta) \equiv 0$ modulo p_1. From the last congruence follows

(11) $$\alpha^m - \beta^m \equiv 0 \pmod{p_1},$$

for $Q_m(\alpha, \beta)$ is a factor of $\alpha^m - \beta^m$. But since

$$\alpha^{m/q} - \beta^{m/q} \text{ is not} \equiv 0 \pmod{p_1},$$

where q is any prime divisor of m, it follows that m is the least exponent for which congruence (11) holds.

These considerations lead to the following theorem:

THEOREM IV. *If α and β are relatively prime integers and $c = mp_1^{a_1}$, where $m > 1$ and not divisible by p_1, then $Q_c(\alpha, \beta)$ is divisible by p_1 if and only if the congruence $\alpha^m \equiv \beta^m \pmod{p_1}$ holds for m and for no exponent less than m. In all other other cases $Q_c(\alpha, \beta) \equiv 1 \pmod{p_1}$.*

If m contains any prime factor greater than p_1, the condition that $\alpha^m \equiv \beta^m \pmod{p_1}$ holds for no exponent less than m is not satisfied; and therefore we have the following corollary:

COROLLARY I. *No one of the prime factors of c except the greatest can divide $Q_c(\alpha, \beta)$. For other prime factors of c we have always $Q_c(\alpha, \beta) \equiv 1$.*

COROLLARY II. *When c contains an odd factor $Q_c(\alpha, \beta)$ is odd, α and β being relatively prime integers.*

If we set $v = p_1^{a_1}$, we may write

$$Q_v(\alpha, \beta) = (\alpha^{p_1 a_1} - \beta^{p_1 a_1}) \div (\alpha^{p_1 a_1 - 1} - \beta^{p_1 a_1 - 1}) \equiv \frac{\alpha - \beta}{\alpha - \beta} \pmod{p_1}.$$

Hence $Q_v(\alpha, \beta) \equiv 1 \pmod{p_1}$ unless $\alpha \equiv \beta \pmod{p_1}$. In the latter case it is evident that $Q_v(\alpha, \beta) \equiv 0 \pmod{p_1}$. Therefore

THEOREM V. *If α and β are relatively prime integers and $v = p_1^{a_1}$,*

$Q_r(\alpha, \beta) \equiv 0$ *or* $1 \pmod{p_1}$ *according as* $\alpha - \beta$ *is or is not congruent to* $0 \pmod{p_1}$.

COROLLARY I. *If* $\alpha \equiv \beta \pmod{p_1}$, $\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}}$ *is divisible by* $p_1^{a_1+1}$.

For,

$$\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}} = \Pi\, Q_d(\alpha, \beta),$$

where d runs over all the a_1+1 divisors of $p_1^{a_1}$. Each factor of the second member contains p_1.

If p_1 is an odd prime the preceding corollary can be made more exact in view of the last clause in Theorem II:

COROLLARY II. *If* p_1^r *is the highest power of an odd prime* p_1 *contained in* $\alpha - \beta$, *then* $\alpha^{p_1^{a_1}} - \beta^{p_1^{a_1}}$ *is divisible by* $p_1^{a_1+r}$ *but by no higher power of* p_1.

Combining Corollary II with Theorem IV we readily deduce the following.

THEOREM VI. *If* α *and* β *are relatively prime integers and* $\alpha - \beta$ *contains* p_1^r *and no higher power of* p_1, *then* $\alpha^{mp_1^{a_1}} - \beta^{mp_1^{a_1}}$ *contains* $p_1^{a_1+r}$ *and no higher power of* p_1, m *as before being prime to* p_1.

We are now prepared to prove the following theorem:

THEOREM VII. *If* α *and* β *are relatively prime integers,* $Q_c(\alpha, \beta)$ *has a prime factor not dividing* $\alpha^s - \beta^s$ $(s<c)$, *except in the cases*

1) $c=2$, $\beta=1$, $\alpha=2^k-1$, *where* k *is an integer.*

2) $Q_c(\alpha, \beta)=p$, *the greatest prime factor of* c, *and* $\alpha^{n/p} - \beta^{n/p} \equiv 0 \pmod{p}$.

3) $Q_c(\alpha, \beta)=1$.*

Every prime factor of $Q_c(\alpha, \beta)$ is evidently contained in $\alpha^c - \beta^c$. Now if $\alpha^s - \beta^s$ and $\alpha^c - \beta^c$ contain a common prime factor q, it follows from the congruences

$$\alpha^s \equiv \beta^s,\ \alpha^c \equiv \beta^c \pmod{q}$$

that

$$\alpha^l \equiv \beta^l \pmod{q},$$

where l is the greatest common divisor of s and c. Hence every common prime factor of $\alpha^s - \beta^s$ and $Q_c(\alpha, \beta)$ is contained in some $\alpha^d - \beta^d$ where d is a divisor of n. Now evidently, every $\alpha^d - \beta^d$ is contained in $\alpha^{c/p_i} - \beta^{c/p_i}$ where for p_i is put the different prime factors of n. By Corollary I to Theorem IV, $Q_c(\alpha, \beta)$ does not contain p unless p is the greatest prime factor of c. Then if $\alpha^{c/p} - \beta^{c/p}$ and $Q_c(\alpha, \beta)$ contain the common factor p, $Q_c(\alpha, \beta)$ does not contain p^2 unless $n=2$, as is seen from Theorems II and III. If $c=2$, $Q_c(\alpha, \beta)=\alpha+\beta$, and this contains a factor different from those of $\alpha-\beta$ unless $\beta=1$, $\alpha=2^k-1$. This accounts for the first exception.

If $n \neq 2$ and $Q_c(\alpha, \beta)$ and $\alpha^{c/p} - \beta^{c/p}$ contain the common factor p,

*The author knows no numerical case for positive β under exception 3) and only one under exception 2); namely, $Q_7(2, 1)=2^3-2+1=3$.

$Q_c(\alpha, \beta)$ does not contain p^2, as we have just seen. But these two numbers contain no other factor in common, as is seen from Theorem I. Hence $Q_c(\alpha, \beta)$ contains a prime not in $\alpha^{c/p}-\beta^{c/p}$ at least in every case for which $Q_c(\alpha, \beta) \neq p$ and also in every case for which $\alpha^{c/p}-\beta^{c/p}$ is not divisible by p, unless $Q_c(\alpha, \beta)=1$.

COROLLARY. *$\alpha^c-\beta^c$ has always a prime factor not dividing $\alpha^s-\beta^s$ $(s<c)$ except in the cases mentioned in Theorem VII.*

NOTE ON THE EXTENSION OF THE EXPONENTIAL THEOREM.

By E. D. ROE, JR., Syracuse University.

In the writer's paper in the June-July number of the MONTHLY, pp. 101-106, it will be observed that the complex roots, infinite in number, arising from the application of an incommensurable exponent are tacitly neglected, only the single real root being used. In fact this is the only root that is of much practical value. It is the only root that is usually considered in the extension of the binomial theorem for the expansion of $(1+x)^n$, even for a commensurable fractional exponent. The finite number of complex roots can be easily expressed if wanted. But when the exponent is incommensurable the number of complex roots becomes infinite and the complex roots become indeterminate.

In both cases, vis., of the binomial theorem, and the exponential theorem, which depends on the binomial theorem, the developments already obtained would have to be multiplied by $\cos\varphi+i\sin\varphi$, to obtain the complete development, where φ admits the value of zero, and has besides an infinite number of values which are indeterminate, when the exponent is incommensurable. All the values would have the same modulus.

These facts are so obvious that it will doubtless appear superfluous to many readers to call attention to them; yet for others it may not be amiss to do so.

DEPARTMENTS.

SOLUTIONS OF PROBLEMS.

ALGEBRA.

319. Proposed by C. N. SCHMALL, New York City.

A man desires to purchase eggs at 5 cents, 1 cent, and $\frac{1}{2}$ cent, respectively, in such numbers that he will obtain 100 eggs for a dollar. How many solutions in rational integers?

I. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.; J. W. CLAWSON, Ursinus College, Collegeville, Pa.; V. M. SPUNAR, Pittsburg, Pa.; and H. C. FEEMSTER, York College, York, Neb.

Let x=number at 5 cents, y=number at 1 cent, z=number at $\frac{1}{2}$ cent.
Then $x+y+z=100=5x+y+\frac{1}{2}z$... (1, 2).
Eliminating z we get the indeterminate equation, $9x+y=100$.
$\therefore y=100-9x$.
This equation gives us eleven integral solutions, as follows:
x=1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11.
y=91, 82, 73, 64, 55, 46, 37, 28, 19, 10, 1.
z=8, 16, 24, 32, 40, 48, 56, 64, 72, 80, 88.

Also solved by S. F. Norris, S. A. Corey, B. Kramer, J. Scheffer, T. J. Fitzpatrick, and Theodore L. DeLand.

II. Solution by PROFESSOR S. F. NORRIS, Baltimore City College, Baltimore, Md., and J. E. SANDERS, Weather Bureau, Chicago, Ill.

$$\text{Average price}=1\left\{\begin{matrix}5\\1\\\frac{1}{2}\end{matrix}\right\}\left|\begin{matrix}1&2&3&4&5&6&7&8&9&10&11\\91&82&73&64&55&46&37&28&19&10&1\\8&16&24&32&40&48&56&64&72&80&88\end{matrix}\right|.$$

NOTE. This is the method used in many arithmetics. For the process of reasoning see *e. g.*, Ray's *New Higher Arithmetic*, p. 333, subject, Alligation. ED. F.

320. Proposed by FRANCIS RUST, C. E., Pittsburg, Pa.

Solve for t, $\cos t=m\cos 2t$.

Solution by T. J. FITZPATRICK, Lamoni, Iowa; S. A. COREY, Hiteman, Iowa; J. E. SANDERS, Weather Bureau, Chicago, Ill.; and B. KRAMER, E. M., N. S., Pittsburg, Pa.

$$\cos t=m\cos 2t=m(\cos^2 t-\sin^2 t)=m(\cos^2 t-1+\cos^2)=2m\cos^2 t-m.$$

$$2m\cos^2 t-\cos t=m.$$

$$\cos^2 t-\frac{1}{2m}\cos t=\tfrac{1}{2}.$$

$$\cos^2 t-\frac{1}{2m}\cos t+\frac{1}{16m^2}=\tfrac{1}{2}+\frac{1}{16m^2}=\frac{8m^2+1}{16m^2}.$$

$$\cos t-\frac{1}{4m}=\pm\frac{1}{4m}\sqrt{[(8m^2+1)]}.$$

$$\cos t = \frac{1}{4m} \pm \frac{1}{4m}\sqrt{[(8m^2+1)]} = \frac{1}{4m}\{1 \pm \sqrt{[(8m^2+1)]}\}.$$

$$\therefore t = \cos^{-1}\{\frac{1}{4m}[1 \pm \sqrt{(8m^2+1)}]\}.$$

Also solved by G. B. M. Zerr, and H. C. Feemster.

GEOMETRY.

344. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

A tinsmith has a sheet of copper in the form of a rectangle, sides a and b. He desires to cut this into two pieces which will form a square when placed together. How can he do this?

I. Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Assuming the base b of the rectangle to be divided into n and the other side a into $n-1$ equal parts. Then, cutting the whole figure by the broken line $m, n, m', n', m'', n'', \ldots$, pushing the left part one step down and shove it one step to the right, we will have for a square,

$$(n-1)x = ny.$$

Also, $a = (n-1)x$, and $b = ny$.

$$\therefore b : a = \left(\frac{n}{n-1}\right)^2.$$

As n must be a positive integer, the relation $b : a$ is *restricted* by the last equation. It is satisfied by the following series of numerical values.

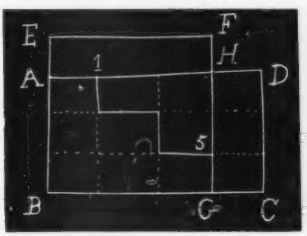
For $n = 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \ldots, \infty$.

$b : a = \infty, 4, 2\frac{1}{4}, 1\frac{7}{9}, 1\frac{9}{16}, 1\frac{11}{25}, 1\frac{13}{36}, 1\frac{15}{49}, 1\frac{17}{64}, 1\frac{19}{81}, \ldots, 1.$

Hence, the required division is possible when the sides of the rectangle are in the ratio 4 : 1.

Also solvee by S. Lefschetz.

II. Solution by C. N. SCHMALL, 604 East 5th Street, New York.



Let $ABCD$ be the given rectangular sheet of copper; $AB = a$, $BC = b$. Let us suppose a square (of paper) $EBGF$ to be constructed by the usual method of the mean-proportional. Then the rectangles $AHFE$ and $GCDH$ are equivalent, and their sides are therefore reciprocally proportional.

$\therefore AH : HD = GH : HF$, or $BG : GC = AB : AE$,

$$\text{or } \frac{BG}{GC}=\frac{AB}{AE}\ldots(1).$$

Hence, if BG be a multiple of GC then AB will be a multiple of AE. In that case divide BG by GC and AB by AE. Now cut along the line $A1$, 2 3, 4 5 to G and then bring 2 3, into coincidence with $A1$, and GC into coincidence with 4 5. Then the given rectangle has been transformed into a square. This problem is impossible if the ratios in equation (1) are not integers. In other words, if $AB=a$, and $BC=b$, $EB=\sqrt{(ab)}$, $EA=\sqrt{(ab)}-a$, $GC=b-\sqrt{(ab)}$, the equation (1) becomes

$$\frac{\sqrt{(ab)}}{b-\sqrt{(ab)}}=\frac{a}{\sqrt{(ab)}-a}=\text{an integer}\ldots(2),$$

and the desired section cannot be performed unless (2) be true.

Assume $\dfrac{a}{\sqrt{(ab)}-a}=\dfrac{\sqrt{a}}{\sqrt{(b)}-\sqrt{(a)}}=i$, an integer.

Then $\sqrt{(b)}=\sqrt{(a)}\,(1+\dfrac{1}{i})$, which is satisfied when $i=1$ only.

345. Proposed by LLOYD HOLSINGER, Bradley Polytechnic Institute, Peoria, Ill.

If a variable polygon move in such a way that its n sides turn severally round n fixed points O_1, O_2, ..., O_n while $n-1$ of its vertices slide, respectively, along $n-1$ fixed straight lines v_1, v_2, ..., v_{n-1}, then the last vertex will describe a conic; and the locus of the point of intersection of any pair of non-adjacent sides will also be a conic. Cremona's *Projective Geometry*.

No solution of this problem has been received.

346. Proposed by G. I. HOPKINS, M. A., Professor of Mathematics and Astronomy, High School, Manchester, N. H.

Prove the theorem for finding the lateral area of a frustum of a cone without the use of the theory of limits.

I. Solution by the PROPOSER.

The theorem relating to the lateral area of the frustum of a cone is given in most text books on Solid Geometry, and proved by the theorem of limits. The accompanying proof does not use limits. The lateral area of the frustum of a cone is equal to its slant height multiplied by one half the sum of the circumferences of its bases.

Let $BADE$ be the frustum of the cone OAD. Designate the radii of its bases by R and r, their circumferences by C and c. Also designate the lateral areas of the two cones by L and l, and their slant heights by S and s.

Then $S-s$ represents the slant height of the frustum, and $L-l$ its lateral area.

$L=\frac{CS}{2}$ and $l=\frac{cs}{2}$. $\therefore L-l=\frac{CS-cs}{2}$.

$\frac{S}{s}=\frac{R}{r}=\frac{C}{c}$. $\therefore Sc=Cs$.

$\therefore Sc-Cs=0$. $CS-cs=CS-cs$.

$\therefore CS+Sc-Cs-cs=CS-cs$. $\therefore (C+c)(S-s)=CS-cs$.

$\therefore L-l=\frac{(S-s)(C+c)}{2}$. Q. E. D.

II. Solution by J. SCHEFFER, A. M., Hagerstown, Md.

Complete the frustum to a full cone, and by cutting along an element spread out the surface into a plane, which can be done with all developable surfaces. The circular arc AB is equal to the circumference of one of the base circles of the frustum$=2\pi R$, and arc $CD=2\pi r$.

$AC=BD=$slant height l of frustum.

From $OA : OC=AB : CD=R : r$, and $OA-OC=l$, we get $OA=\frac{Rl}{R-r}$, $OC=\frac{rl}{R-r}$.

Area of $ABCD=\frac{1}{2}AB.OA-\frac{1}{2}CD.OC=\frac{1}{2}.2\pi R.\frac{Rl}{R-r}-\frac{1}{2}.2\pi r.\frac{rl}{R-r}=$ $\pi l.\frac{R^2-r^2}{R-r}=\pi l(R+r)$.

CALCULUS.

277. Proposed by W. J. GREENSTREET, M. A., Marling School, Stroud, England.

Find $\frac{d^2x}{ds^2}$ and $\frac{d^2y}{ds^2}$ for $y=c\sinh\frac{x}{c}$.

Solution by J, W. CLAWSON, Collegeville, Pa.

$y=c\sinh\frac{x}{c}$. $\therefore \frac{dy}{dx}=\cosh\frac{x}{c}$. But $ds^2=dx^2+dy^2$.

$\therefore \left(\frac{ds}{dx}\right)^2=1+\cosh^2\frac{x}{c}$ and $\left(\frac{ds}{dy}\right)^2=1+\frac{1}{\cosh^2(x/c)}$.

$\therefore \frac{dx}{ds}=(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}$ and $\frac{dy}{ds}=(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}.\cosh\frac{x}{c}$.

$\therefore \frac{d^2x}{ds^2}=-\frac{1}{2}(1+\cosh^2\frac{x}{c})^{-\frac{3}{2}}.2\cosh\frac{x}{c}\sinh\frac{x}{c}.\frac{1}{c}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}$

$$=-\frac{1}{c}\frac{\sinh(x/c)\cosh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

$$\frac{d^2y}{ds^2}=-\tfrac{1}{2}(1+\cosh^2\frac{x}{c})^{-\frac{3}{2}}.2\cosh\frac{x}{c}\sinh\frac{x}{c}.\frac{1}{c}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}.\cosh\frac{x}{c}$$

$$+\sinh\frac{x}{c}.\frac{1}{c}(1+\cos^2\frac{x}{c})^{-\frac{1}{2}}.(1+\cosh^2\frac{x}{c})^{-\frac{1}{2}}=\frac{1}{c}\frac{\sinh(x/c)}{[1+\cosh^2(x/c)]^2}.$$

Also solved by H. C. Feemster, G. B. M. Zerr, V. M. Spunar, and J. Scheffer.

278. Proposed by S. A. COREY, Hiteman, Iowa.

If C be Euler's constant, .577,215,664,9... and if B_1, B_2, B_3, etc., be Bernoulli's numbers, $\frac{1}{6}$, $\frac{1}{30}$, $\frac{1}{42}$, etc., prove that

$$C=\tfrac{1}{2}+\frac{B_1}{2}-\frac{B_2}{4}+\frac{B_3}{6}-\frac{B_4}{8}+\ldots-(1)^m\frac{B_m}{2m}+\ldots.$$

I. Solution by V. M. SPUNAR, M. and E. E., East Pittsburg, Pa.

Euler's constant C may be presented under various forms from among which an elementary one may be the following:

$$C=\underset{n=\infty}{\text{Lim.}}\,[1+\tfrac{1}{2}+\tfrac{1}{3}+\tfrac{1}{4}+\ldots\frac{1}{n}-\log n]=.577,215,664,9\ldots$$

By Taylor's Theorem, we have

$$f(a+h)-f(a)=hf'(a)+\frac{h^2}{2!}f''(a)+\frac{h^3}{3!}f'''(a)+\ldots+\frac{h^n}{n!}f^{(n)}(a)+\ldots$$

Change a successively into $a+h$, $a+2h$, ..., $a+(n-1)h$ and add, then if we put x for $(a+nh)$, we obtain the following result:

$$f(x)-f(a)=h\,\Sigma f'(x)+\frac{h^2}{2!}\Sigma f''(x)+\frac{h^3}{3!}\Sigma f'''(x)+\ldots+\frac{h^3}{n!}\Sigma f^{(n)}(x)+\ldots$$

where $\Sigma f'(x)=f'(x)+f'(a+h)+\ldots$, $\Sigma f''(x)=f''(a)+f''(a+h)+\ldots$, etc.

Let $\phi(x)=f'(x)$. Then

$$\int_a^{a+nh}\phi(x)dx=h\,\Sigma\,\phi(x)+\frac{h^2}{2!}\Sigma\,\phi'(x)+\frac{h^3}{3!}\Sigma\,\phi''(x)+\ldots+\frac{h^n}{n!}\Sigma\,\phi^{(n-1)}(x)+\ldots.$$

$$\therefore \Sigma\,\phi\,(x)=\frac{1}{h}\int_a^{a+nh}\phi(x)dx-\frac{h}{2\,!}\,\Sigma\,\phi'(x)\,-\frac{h^2}{3\,!}\,\Sigma\,\phi''(x)-\ldots$$

$$-\frac{h^n}{(n+1)\,!}\,\Sigma\,\phi^{(n)}(x)-\ldots(1).$$

Similarly, $\Sigma\,\phi'(x)=\frac{1}{h}[\phi(x)-\phi(a)]-\frac{h}{2\,!}\,\Sigma\,\phi''(x)-\frac{h^2}{3\,!}\,\Sigma\,\phi'''(x)-\ldots$

$$-\frac{h^n}{(n+1)\,!}\phi^{(n+1)}(x)-\ldots(2).$$

Now from the series (1) we may eliminate $\Sigma\,\phi'(x)$, $\Sigma\,\phi''(x)$, ..., by the aid of (2), (3), ... In order to do it multiply (2) by A_0h, (3) by A_1h^2, ..., then add the results and determine A_0, A_1, A_2, ..., by equations

$$A_0+\frac{1}{2\,!}=0.$$

$$A_1+\frac{A_0}{2\,!}+\frac{1}{3\,!}=0.$$

$$A_2+\frac{A_1}{2\,!}+\frac{A_0}{3\,!}+\frac{1}{4\,!}=0.$$

. . . . .

$$\therefore \Sigma\,\phi(x)\,=\frac{1}{h}\int_a^{a+nh}\phi(x)dx+A_0[\phi(x)-\phi(a)]+A_1[\phi'(x)-\phi'(a)]h$$

$$+A_2[\phi''(x)-\phi''(a)]h^2+\ldots$$

As it has been shown that $\Sigma\,\phi(x)$ can be put in this form, where A_0, A_1, A_2, ..., A_n, ... are numerical quantities independent of the variable x and of the function $\phi(x)$, viz:

$$\Sigma\,\phi(x)\,=\frac{x}{e^x-1}=1-\frac{x}{2}+\frac{B_1}{2\,!}x^2\,-\frac{B_2}{4\,!}x^4+\ldots\,(-1)^{m+1}\frac{B_m}{2m\,!}+\ldots,\text{ where}$$

$A_m=(-1)^{m+1}\frac{B_m}{2m\,!}$, and B_1, B_2, B_3, ..., B_m are the well known Bernoulli's numbers ($A_0=-\frac{1}{2}$).

$$\therefore \Sigma\,\phi(x) = \frac{1}{h}\int_a^x \phi(x)dx - \tfrac{1}{2}[\phi(x)-\phi(a)] + \frac{B_1}{2\,!}[\phi'(x)-\phi'(a)]h$$

$$- \frac{B_2}{4\,!}[\phi'''(x)-\phi'''(a)]h^3 + \ldots$$

Also,

$$\therefore \Sigma\,\phi(x) = C + \frac{1}{h}\int \phi(x)dx - \tfrac{1}{2}\phi(x) + \frac{B_1}{2}\phi'(x)\frac{h}{1\,!} - \frac{B_2}{4}\phi'''(x)\,\frac{h^3}{3\,!} + \ldots$$

$$\ldots(-1)^{m+1}\frac{B_m}{2_m}\phi^{(2m-1)}(x)\,\frac{h^{2m-1}}{(2m-1)\,!} + \ldots (M),$$

where C represents a series of terms independent of x.

Taking $\phi(x)=1/x$ and $h=1$ then we get, adding $(1/x)$ to both sides,

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\ldots\frac{1}{x} = C + \log x + \frac{1}{2x} - \frac{1}{12x^2} + \ldots$$

Hence, by making $x=\infty$ we infer that in this example C is Euler's constant (according to our definition).

But on the other hand in (M),

$$C = -\frac{1}{h}\int \phi(a)da + \tfrac{1}{2}\phi(a) + \frac{B_1}{2}\phi'(a)\frac{h}{1\,!} - \frac{B_2}{4}\,\phi'''(a)\,\frac{h_3}{3\,!} +$$

$$\ldots(-1)^{m+1}\frac{B_m}{2m}\phi^{(2m-1)}(a)\,\frac{h^{2m-1}}{(2m-1)\,!}\ldots(N).$$

Hence, put $\phi(a)=1/a$, $h=1$ (as above), and then $a=1$.

$$\therefore C \equiv \tfrac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \ldots(-1)^{m+1}\frac{B_m}{2m}\ldots \quad \text{Q. E. D.}$$

II. Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Taking $a=1$, $x=m$ in the development in problem 237, Calculus, we get

$$\log(1+m) = 0 + \tfrac{1}{2}\left[\frac{1}{1+m}+1+2\left(\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\ldots+\frac{1}{m}\right)\right]$$

$$+\frac{B_1}{2\,!}\left[\frac{1}{(1+m)^2}-1\right] - \frac{B_2}{4\,!}\left[\frac{1}{(1+m)^4}-3!\right] + \text{etc.}$$

Let m approach ∞. Then

$$\lim_{n=\infty} \log n = \left[\tfrac{1}{2} + \sum_{p=2}^{p=\infty} \frac{1}{p}\right] - \left[\frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \ldots\right]$$

But $\lim_{n=\infty} \log n - \sum_{p=2}^{p=\infty} \frac{1}{p} = -C+1.$

$$\therefore -C+1 = \tfrac{1}{2} - \frac{B_1}{2} + \frac{B_2}{4} - \frac{B_3}{6} + \frac{B_4}{8} - \ldots$$

$$\therefore C = \tfrac{1}{2} + \frac{B_1}{2} - \frac{B_2}{4} + \frac{B_3}{6} - \frac{B_4}{8} + \ldots.$$

Also solved by the Proposer.

MECHANICS.

227. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

Regarding the earth as a homogeneous sphere, radius R, acceleration at the surface g, investigate the motion of a sphere, radius b, moving through a straight tunnel between two points on the surface not diametrically opposite.

II. Solution by the PROPOSER.

Let AB be the tunnel; C its mid-point; D the position of the sphere at any time t; O the center of the earth; $OD=y$, $AC=a$, $DC=x$, $\angle DOC=\theta$; f' the acceleration at D. Then $f' : y = g : R$ or $f' = gy/R$. But $y = x/\sin\theta$; also $f = f' \sin\theta$. $\therefore f = gx/R$.

$$\text{Now } \int_{v_1}^{v} v\,dv = f\int_{x}^{a} dx = \frac{g}{R}\int_{x}^{a} x\,dx.$$

$$\therefore v^2 = (dx/dt)^2 = g/R(a^2 - x^2) + v_1^2.$$

$$\therefore t = \sqrt{(R/g)}\int_0^a \frac{dx}{\sqrt{[Rv_1^2 + g(a^2 - x^2)]}}.$$

$$\therefore t = (R/g)\sin^{-1}\frac{a\sqrt{g}}{\sqrt{[Rv_1^2 + ga^2]}}.$$

If the sphere starts from rest, $v_1 = 0$.

$\therefore t = (\pi/2)\sqrt{(R/g)}$.

If the tunnel is perfectly rough and F is the friction, we get for the equations of motion: $d^2x/dt^2 + gx/R + F = 0$; $k^2(d^2\phi/dt^2) = bF$; $bd\phi = dx$.

$$\therefore \frac{d^2x}{dt^2}\left(\frac{b^2+k^2}{b^2}\right)+\frac{gx}{R}=0. \quad \therefore \frac{dx^2}{dt^2}+\frac{5gx}{7R}=0.$$

$$\left(\frac{dx}{dt}\right)^2=\frac{5g}{7R}(a^2-x)^2 \text{ or } T=\frac{\pi}{2}\sqrt{\frac{7R}{5g}}.$$

Let R=20902410 feet, g=32.10614 feet. Then $\sqrt{(R/g)}$=806.871.
t=1267.433 seconds=21 minutes, 7.433 seconds.
T=1499.627 seconds=24 minutes, 59.627 seconds.

t, T the time for the sphere to move from rest at the surface to the middle of the tunnel is constant for tunnels of all lengths and the same as for a tunnel passing through the center of the earth.

228. Proposed by J. E. ROSE, Mount Angel College, Mount Angel, Oregon.

AB, *BC* are two uniform rods freely hinged at *B*, whose weights are W, $4W$, and lengths $2a$, $4a$, respectively. The ends *A*, *C* of the rods are attached to small rings which slide on a rough horizontal wire. When the distance between the rings is the greatest for which equilibrium can exist, both of them are on the point of slipping. Find the coefficient of friction.

Solution by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa., and S. LEFSEHETZ, Pittsburg, Pa.

Let R, R' be the reactions at C, A, respectively; μR, $\nu R'$ the friction at these points; $\angle BAC=\theta$, $\angle BCA=\phi$.
Then $R+R'=5W$, $\mu R=\nu R'$... (1, 2).
Taking moments around B we get

$2R'\cos\theta=2\nu R'\sin\theta+W\cos\theta$ or $R'=W/[2(1-\nu\tan\theta)]$... (3).
$2R\cos\phi=2\mu R\sin\phi+4W\cos\phi$ or $R=2W/(1-\mu\tan\phi)$... (4).

(1, 2) in (3) and (4) gives $9\mu-\nu=10\mu\nu\tan\theta$, $3\nu-2\mu=5\mu\nu\tan\phi$.

$$\therefore \frac{5}{\tan\phi+6\tan\phi}=\nu, \quad \frac{5}{4\tan\theta+9\tan\phi}=\mu.$$

In this problem, $\mu=\nu$. $\therefore \mu=\frac{1}{5}\cot\theta=\frac{1}{5}\cot\phi$. $\therefore \tan\theta=4\tan\phi$.
Also $\sin\theta : \sin\phi=2 : 1$, or $\sin\theta=2\sin\phi$.
$\therefore \tan\phi=\frac{1}{2}$. $\therefore \mu=\frac{2}{5}$.

PROBLEMS FOR SOLUTION.

ALGEBRA.

324. Proposed by R. D. CARMICHAEL, Princeton University.

Sum the *finite* series

$$\frac{16n^2-2^2}{4!}-\frac{(16n^2-2^2)(16n^2-4^2)}{6!}+\frac{(16n^2-2^2)(16n^2-4^2)(16n^2-6^2)}{8!}-\ldots$$

where n is a positive integer.

325. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

I have a chronometer whose rate is uniform. When it indicates t_1 time at Washington I find that it is h_1 hours slow. I take it to Philadelphia and when it indicates t_2 time, the local time of Philadelphia is h_2 hours faster. I bring my chronometer back to Washington and find that when it indicates t_3 time it is h_3 hours slow. If $t_1 = 5$ A. M., $t_2 = 7$ hours, 54 minutes, $t_3 = 11$ hours, 46 minutes A. M., $h_1 = 1$ hour, $h_2 = 1\ 203/900$ hours, $h_3 = 1\ 7/30$ hours, find the difference of longitude between Washington and Philadelphia.

GEOMETRY.

351. Proposed by L. E. DICKSON, Ph. D., The University of Chicago.

Given an isosceles right triangle with hypothenuse h; an isosceles triangle with two sides h and two angles $A=22^\circ\ 30'$; a right angle triangle with the same angle A and opposite side $h/\sqrt{2}$; a triangle with the same angle A, opposite side h, and an angle 45°. Form a triangle whose four pieces are these four triangles, and prove geometrically that it is isosceles.

352. Proposed by G. I. HOPKINS, Professor of Astronomy, High School, Manchester, N. H.

Required, to construct the triangle, having given the base, vertical angle and sum of the altitude and the two remaining sides.

353. Proposed by L. H. McDONALD, M. A., Ph. D., Sometimes Tutor at Cambridge, Jersey City, N. J.

In a given circle place two chords which shall be in a given ratio and also a given distance apart.

CALCULUS.

282. Proposed by S. G. BARTON, Ph. D., Clarkson School of Technology, Potsdam, N. Y.

A rectangular beam of length l and width w is taken horizontally from a hall of width b into a corridor at right angles to the hall. Find the width of the smallest corridor into which it can be taken.

283. Proposed by B. F. FINKEL, Ph. D., Drury College.

By means of the calculus, determine the angle of minimum deviation of a ray of monochromatic light in passing through a triangular prism.

MECHANICS.

236. Proposed by G. B. M. ZERR, A. M., Ph. D., Philadelphia, Pa.

A simple beam length $2a$, supported at the ends, is loaded with c pounds per running foot at the ends and increases uniformly to the center, where it is b pounds per running foot. Find deflection at center due to this load.

237. Proposed by C. N. SCHMALL, 604 East 5th Street, New York.

In a naval action an officer observes that in the case of two guns firing, at elevations α and β, respectively, the projectile of the former falls a feet short of the target while that of the latter lands b feet beyond. The initial velocity being the same in both cases, prove that the *true* elevation is

$$\tfrac{1}{2}\sin^{-1}\left[\frac{a\sin 2\beta + b\sin 2\alpha}{a+b}\right].$$

(Suggested by problem 29, page 219, Jeans' *Theoretical Mechanics.*)

NUMBER THEORY AND DIOPHANTINE ANALYSIS.

168. Proposed by A. H. HOLMES, Brunswick Maine.

Find integral values for x, y, u and v from the following:

$$uv - xy = 25x + 29y + 29u + 29v - 112.$$
$$3v - 5u + 5y - x = 102.$$
$$4y - 3v = 419.$$

169. Proposed by R. D. CARMICHAEL, Princeton University.

Let $Q_n(x) = 0$ be the equation whose roots are all the primitive nth roots of unity without repetition. In $Q_n(x) = 0$ replace x by α/β, a fraction in its lowest terms, and clear of fractions. Let $Q_n(\alpha, \beta)$ represent the resulting first member. Set $n = mp$ where p is the largest prime factor of n. It is required to find all the integral values of α, β, m, p satisfying the following relations:

(1) $Q_m p(\alpha, \beta) = p$,

(2) $\alpha^m - \beta^m \equiv 0 \pmod{p}$.

One such solution is: $\alpha = 2$, $\beta = 1$, $m = 2$, $p = 3$. (See MONTHLY, Vol. XII, p. 89.)

NOTES AND NEWS.

The next annual meeting of the American Association for the Advancement of Science will be held at Boston during the week beginning December 27. Titles and abstracts of papers intended for the section of Mathematics and Astronomy should reach Professor G. A. Miller, 907 West Nevada Street, Urbana, Ill., before December 15.

Professor G. H. Scott of Yankton College, S. D., is spending the present year at the University of Illinois as assistant in mathematics, and graduate student. M.

The American Mathematical Society decided to contribute 5000 francs towards the publication of the complete works of the noted Swiss mathematician, Leonard Euler. M

The following promotions and appointments for the present year to positions in Mathematics in colleges and universities will be of interest to the readers of the MONTHLY:

R. D. Carmichael to a teaching fellowship at Princeton University.

Dr. N. J. Lennes to an instructorship in the Massachusetts Institute of Technology.

Dr. F. L. Griffin to an assistant professorship at Williams College.

G. P. Paine, of Ripon College, to an assistant professorship at the Univessity of Minnesota.

Dr. W. R. Longley to an assistant professorship at Yale University.

Dr. C. N. Haskins to an associate professorship at Dartmouth College.

T. Hildebrandt to an instructorship at the University of Michigan.

Dr. A. L. Underhill to an assistant professorship at the University of Minnesota.

L. L. Silverman to an instructorship at Cornell University.

Dr. G. D. Birkhoff to an assistant professorship at Princeton University.

E. J. Moulton to an instructorship at the University of Wisconsin.

Dr. Thomas Buck to an instructorship at the University of Illinois.

Dr. Arnold Dresden to an instructorship at the University of Wisconsin.

Dr. C. L. E. Moore to an assistant professorship at Massachusetts Institute of Technology.

Dr. H. E. Buchanan to an instructorship at the University of Wisconsin.

Dr. J. H. Maclagan-Wedderburn to a preceptorship at Princeton University. S.

The American Federation of Teachers of the Mathematical and the Natural Sciences, at its meeting in Baltimore, December, 1908, authorized the appointment by its council of a committee of fifteen on a syllabus of geometry for secondary schools. A similar committee had been authorized by the Mathematics Section of the Secondary Department of the National Education Association at its meeting in Cleveland in 1908. It has been dedided that the same committee shall act under the joint auspices of the two national bodies. The membership of the committee is as follows: H. E. Slaught, chairman, Univeasity of Chicago: C. L. Bouton, Harvard Univer-

sity; Florian Cajori, Colorado College; H. E. Hawks, Yale University; E. R. Hardrick, University of Missouri; H. L. Rietz, University of Illinois; D. E. Smith, Teachers College, Columbia University; William Betz, East High School, Rochester, N. Y.; E. L. Brown, North High School, Denver, Colo.; W. B. Carpenter, Mechanics Arts High Suhool, Boston, Mass.; W. W. Hart, Shortridge High School, Indianapolis, Ind.; F. K. Newton, Phillips Andover Academy, Andover, Mass.; E. R. Smith, Brooklyn, N. Y. Polytechnic School; R. L. Short, Technical High School, Cleveland, O.; and Mabel Sykes, South Chicago High School, Chicago, Ill.

The committee is already at work, having resolved itself into three sub-committees of five members each, under the chairmanship of Professors D. E. Smith, E. R. Hedrick, and H. L. Rietz, respectively, for the study of the three phases: Logical considerations, lists of basal theorems, and exercises and applications. S.

Among the important mathematical meetings in the near future are the following:

The Central Association of Science and Mathematics Teachers at the University of Chicago, November 26, 27, 1909.

The American Federation of Teachers of the Mathematical and the Natural Sciences at Boston during the holidays.

The American Mathematical Society, in connection with Section A of the American Association for the Advancement of Science, at Boston during the holidays.

The Chicago Section of the American Mathematical Society at the University of Chicago December 31, 1909, January 1, 1910,

The Southwestern Section of the American Mathematical Society at the University of Missouri in the Thanksgiving recess.

B. F. Finkel will address the Southwest Section of the Missouri State Teachers' Association on November 26, on "The Teaching of Mathematics for College Entrance."

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THE TEACHING OF MATHEMATICS IN THE COLLEGES.

By H. E. SLAUGHT, The University of Chicago.

The first decade of the twentieth century is witnessing a wide spread wave of interest in the better teaching of secondary mathematics. This interest includes both the subject-matter and the form of presentation and arrangement in the curriculum and also the preparation of teachers for secondary schools.

Among the important activities pertaining to the subject-matter and its presentation may be mentioned the large number of associations of teachers, banded together for the purpose of improving the teaching of mathematics, in cities, in states, and in large sections of the country, such as the New England Association, the Middle States and Maryland Association, and the Central Association. In all of these associations, by papers, discussions, syllabi, committee reports, and experience meetings, every phase of secondary mathematical work has undergone the most careful reconsideration.

With respect to the preparation of teachers of secondary subjects, a great change has also taken place in recent years. Whereas, formerly there was no prevalent demand for college graduates as high school teachers, it is now true that it is very difficult for a teacher without a college degree to secure an appointment in any first-class high school. But further than this, specific training in the principles and practice of education is coming to be demanded. A notable instance of this is the state requirement in California, where no teacher can be certificated without a year's work in education in the University of California or in some institution where work of a high grade is offered.

While doubtless this standard of preparation has been set by the better equipped universities and by the few schools or colleges of education, yet the demand for such training is now being reflected back from the schools and compelling colleges and universities all over the country to turn their attention to the departments of education. The writer, in connection with the Board of Recommendations of the University of Chicago, had occasion recently to ask information from about two hundred institutions with